The Union-Closed Sets Conjecture for Small Families

Jens Maßberg *

August 26, 2015

Keywords: union-closed sets, Frankl's conjecture

Abstract

We prove that the union-closed sets conjecture is true for separating union-closed families $\mathcal{A}$ with $|\mathcal{A}| \leq 2\left(m + \frac{m}{\log_2(m) - \log_2\log_2(m)}\right)$ where m denotes the number of elements in $\mathcal{A}$.

1 Introduction

A family $\mathcal{A}$ of sets is said to be *union-closed* if for any two member sets $A, B \in \mathcal{A}$ their union $A \cup B$ is also a member of $\mathcal{A}$.

A well-known conjecture is the *Union-Closed Sets Conjecture* which is also called *Frankl's conjecture*:

Conjecture 1.1. *Any finite non-empty union-closed family of sets has an element that is contained in at least half of its member sets.*

There are many papers considering this conjecture. So it is known to be true if $\mathcal{A}$ has at most 12 elements [8] or at most 50 member sets [4, 7] or if the number of member sets is large compared to the number m of elements, that is $|\mathcal{A}| \geq \frac{2}{3}2^m$ [1]. Nevertheless, the conjecture is still far from being proved or disproved. A good survey on the current state of this conjecture is given by Bruhn and Schaudt [2].

In this paper we consider the case that the number of member-sets is small compared to the number of elements. But first we recall some basic definitions and results. Let $\mathcal{A}$ be a union-closed set. We call $U(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ the *universe* of $\mathcal{A}$. For an element $x \in U(\mathcal{A})$ the cardinality of $|\{A \in \mathcal{A} : x \in A\}|$ is called the *frequency* of x. Thus the union-closed sets conjecture states that there exists an element $x \in U(\mathcal{A})$ of frequency at least $\frac{1}{2}|\mathcal{A}|$.

A family $\mathcal{A}$ is called *separating* if for any two distinct elements $x, y \in U(\mathcal{A})$ there exists a set $A \in \mathcal{A}$ that contains exactly one of the elements x and y. We

*Institut für Optimierung und Operations Research, University of Ulm, jens.massberg@uni-ulm.de

can restrict ourselves to separating union-closed families: If there exist elements x and y such that each member set $A \in \mathcal{A}$ that contains x also contains y, then we can delete x from each such set and obtain a new family of the same cardinality that is still union-closed. Falgas-Ravry showed that there are some sets in $\mathcal{A}$ satisfying certain conditions which help us to analyze small separating union-closed families:

Theorem 1.2 (Falgas-Ravry [3]). *Let $\mathcal{A}$ be a separating union-closed family and let $x_1, \ldots, x_m$ be the elements of $U(\mathcal{A})$ labeled in order of increasing frequency. Then there exist sets $X_0, \ldots, X_m \in \mathcal{A}$ such that*

$$x_i \notin X_i \quad \forall i \in \{1, \ldots, m\} \tag{1}$$

and

$$\{x_{i+1}, \ldots, x_m\} \subset X_i \quad \forall i \in \{0, \ldots, m\} \tag{2}$$

Proof. As $\mathcal{A}$ is separating, for any $1 \leq i < j \leq m$ there exists a set $X_{ij} \in \mathcal{A}$ such that $x_i \notin X_{ij}$ and $x_j \in X_{ij}$. For all $1 \leq i \leq m-1$ let $X_i = \bigcup_{j=i+1}^{m} X_{ij}$ and set $X_0 = U(\mathcal{A})$. □

The previous theorem directly implies that the conjecture is satisfied for small families:

Lemma 1.3. *Any separating family on m elements with at most $2m$ member sets satisfies the Union-Closed Sets Conjecture.*

Proof. Consider the sets $X_0, \ldots, X_{m-1}$ constructed in Theorem 1.2 and observe that the most frequent element x_m is contained in all these sets. As these sets are pairwise different, x_m is contained in at least m of all member sets of $\mathcal{A}$. □

In this paper we show that the Union-Closed Sets Conjecture is also satisfied for families that contain (slightly) more then $2m$ member sets. Considering such families is motivated by a result of Hu (see also [2]):

Theorem 1.4 (Hu [5]). *Suppose there is a $c > 2$ so that any separating union-closed family $\mathcal{A}'$ with $|\mathcal{A}'| \leq c|U(\mathcal{A}')|$ satisfies the Union-Closed Sets Conjecture. Then, for every union-closed family $\mathcal{A}$, there is an element $x \in U(\mathcal{A})$ of frequency*

$$|\{A \in \mathcal{A} : x \in A\}| \geq \frac{c-2}{2(c-1)}|\mathcal{A}|. \tag{3}$$

Therefore, if the Union-Closed Sets Conjecture is satisfied for 'small' families, then for any union-closed family there exists an element that appears with a frequency at least a constant fraction of the number of member sets. In this paper we push the bound over $2m$, but for increasing m it still converges slowly towards $2m$.

2 Frankl's Conjecture for Small Families

Combining and extending the idea of the proof of Theorem 1.2 and an argument of Knill [6] we get the main result of this paper.

Theorem 2.1. *The Union-Closed Sets Conjecture is true for separating union-closed families $\mathcal{A}$ with a universe containing m elements satisfying*

$$|\mathcal{A}| \leq 2\left(m + \frac{m}{\log_2(m) - \log_2\log_2(m)}\right).$$

Proof. Let $\mathcal{A}$ be a separating union-closed family, let the elements $x_1, \dots, x_m$ of $U(\mathcal{A})$ be labeled in order of increasing frequency and set $n = |\mathcal{A}|$. Assume that each element appears in at most $m+c$ member sets. We compute an upper bound on the size of n.

For $i \in \{1, \dots, m\}$ we set

$$M_i = \bigcup_{A \in \mathcal{A}: x_i \notin A} A \tag{4}$$

to be the union of all sets containing x_i and we set $M_0 = U$. If the sets X_i, $i \in \{0, \dots, m\}$, are chosen as in Theorem 1.2, then we have $X_i \subset M_i$ for all $i \in \{0, \dots, m-1\}$ and thus

$$\{x_{i+1}, \dots, x_m\} \subseteq M_i. \tag{5}$$

Let $\tilde{U} = \{x_i : \exists A \in \mathcal{A} \text{ with } \max_{x_j \in A} j\}$ be the set of all x_i which are the elements with the highest index in some set A.

For $x_i \in \tilde{U}$ we set

$$A_i = \bigcup_{A \in \mathcal{A}:\, i = \max\{j:\, x_j \in A\}} A. \tag{6}$$

By definition $x_i \in A_i$. Now consider $j > i$. As $x_j \notin A_i$ we have $A_i \subset M_j$. Together with (5) we have

$$x_i \in M_j \quad \forall x_i \in \tilde{U}, j \in \{0, \dots, m-1\}, i \neq j. \tag{7}$$

Observe that every non-empty member set of $\mathcal{A}$ touches $\tilde{U}$. Following an argument of Knill [6] let $\hat{U} \subseteq \tilde{U}$ be minimal such that every non-empty set of $\mathcal{A}$ touches $\hat{U}$. Then for all $x_i \in \hat{U}$ there exists a set $A \in \mathcal{A}$ with $\hat{U} \cap A = \{x_i\}$; if not, $\hat{U} \setminus \{x_i\}$ still touches every member set of $\mathcal{A}$ contradicting the minimality of $\hat{U}$. Therefore as $\mathcal{A}$ is union-closed, for each $B \subseteq \hat{U}$ there exists a set $P_B \in \mathcal{A}$ with $P_B \cap \hat{U} = B$. Let $\mathcal{P} = \{P_B : B \subseteq \hat{U}\}$. The sets in $\mathcal{P}$ are pairwise disjoint and each element $x_i \in \hat{U}$ is contained in exactly half of the sets. Setting $k = |\hat{U}|$, we conclude that there are 2^k sets in $\mathcal{P}$ containing in total $k2^{k-1}$ elements from $\hat{U}$.

Note, that $\mathcal{P}$ might contain the sets M_i for $x_i \in \hat{U}$ and one additional set M_j with $\hat{U} \subset M_j$. But then $\{M_0, \dots, M_{m-1}\}$ contains $m-k$ sets that are not in $\mathcal{P}$ and each of these sets contains all elements of $\hat{U}$.

Before we compute an upper bound for the number of elements in $\mathcal{A}$ we summarize the previous observations:

- Each of the k elements in $\hat{U}$ appears in at most $m+c$ member sets,
- the 2^k sets in $\mathcal{P}$ contain in total $k2^{k-1}$ copies of elements of $\hat{U}$,
- there are $m-k$ additional member sets, each containing all elements of $\hat{U}$ and
- all remaining member sets contain at least one element of $\hat{U}$.

We conclude:

$$\begin{aligned} n &\leq k(m+c) + (2^k - k2^{k-1}) + (m-k)(1-k) && (8)\\ &= m + kc + (2-k)2^{k-1} + k^2 - k. && (9) \end{aligned}$$

Suppose the Union-Closed Sets Conjecture is wrong, that is, $n > 2(m+c)$ or $\frac{n}{2} - m > c$. Then

$$n \leq m + k(\frac{n}{2} - m) + (2-k)2^{k-1} + k^2 - k \qquad (10)$$

or

$$\begin{aligned} n &\geq 2\frac{(k-1)m + (k-2)2^{k-1} + k - k^2}{k-2} && (11)\\ &\geq 2\left(m + 2^{k-1} + \frac{m}{k-2} - k - 3\right). && (12) \end{aligned}$$

We conclude that the conjecture is true for all n satisfying

$$n \leq 2\left(m + \min_{k\in\mathbb{N}}\left(2^{k-1} + \frac{m}{k-2} - k - 3\right)\right). \qquad (13)$$

The function $f_m(k) := 2^{k-1} + \frac{m}{k-2} - k - 3$ is convex. Živković et al. [8] showed that the Union-Closed Sets Conjecture is satisfied for $m \leq 12$ so we can assume that $m \geq 13$. In this case the minimum of $f_m(k)$ is obtained in the interval $[5, \log_2(m)]$ and we get

$$\begin{aligned} f_m(k) &= \max\left\{2^{k-1}, \frac{m}{k-2}\right\} + \left(\min\left\{2^{k-1}, \frac{m}{k-2}\right\} - 3 - k\right) && (14)\\ &\geq \max\left\{2^{k-1}, \frac{m}{k-2}\right\} && (15)\\ &\geq \min_{k'}\left(\max\left\{2^{k'-1}, \frac{m}{k'-2}\right\}\right) && (16)\\ &\geq \max_{k'}\left(\min\left\{2^{k'-1}, \frac{m}{k'-2}\right\}\right). && (17) \end{aligned}$$

The last inequality is due to the fact that 2^{k-1} is increasing in k while $\frac{m}{k-2}$ is decreasing in k.

Setting $k' = \log_2(m) - \log_2\log_2(m) + 2$ we get

$$\begin{aligned}
\log_2\left(\frac{m}{k'-2}\right) &= \log_2(m) - \log_2\left(\log_2(m) - \log_2\log_2(m)\right) \\
&= \log_2(m) - \log_2\log_2(m) - \log_2\left(1 - \frac{\log_2\log_2(m)}{\log_2(m)}\right) \\
&\leq \log_2(m) - \log_2\log_2(m) + 1 \\
&= \log_2(2^{k'}).
\end{aligned}$$

Inserting this result in (17) and (13) we finally obtain that the Union-Closed Sets Conjecture is true for all n satisfying

$$n \leq 2\left(m + \frac{m}{\log_2(m) - \log_2\log_2(m)}\right). \tag{18}$$

□

3 Acknowledgement

The author thanks Henning Bruhn-Fujimoto for pointing him to the union-closed sets conjecture.

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